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A NECESSARY AND A SUFFICIENT CONDITION FOR THE EXISTENCE OF THE POSITIVE RADIAL SOLUTIONS TO HESSIAN EQUATIONS AND SYSTEMS WITH WEIGHTS

DRAGOS-PATRU COVEI

Abstract. In this article we consider the existence of positive radial solutions for Hessian equations and systems with weights and we give a necessary condition as well as a sufficient condition for a positive radial solution to be large. The method of proving theorems is essentially based on a successive approximation. Our results complete and improve a recently work published by Zhang and Zhou (*Existence of entire positive k-convex radial solutions to Hessian equations and systems with weights*, Applied Mathematics Letters, Volume 50, December 2015, Pages 48–55).

Contents

1. Introduction

Let D^2u be the Hessian matrix of a C^2 (i.e., a twice continuously differentiable) function u defined over $\mathbb{R}^N$ $(N \geqslant 3)$ and $\lambda\left(D^2u\right) = (\lambda_1, ..., \lambda_N)$ the vector of eigenvalues of D^2u. For $k = 1, 2, ..., N$ is defined the k-Hessian operator as follows

$$S_k\left(\lambda\left(D^2u\right)\right) = \sum_{1\leqslant i_1<...<i_k\leqslant N} \lambda_{i_1}\cdot ... \cdot \lambda_{i_k}$$

i.e., it is the k^{th} elementary symmetric polynomial of the Hessian matrix of u. In other words, $S_k\left(\lambda\left(D^2u\right)\right)$ it is the sum of all $k \times k$ principal minors of the Hessian matrix D^2u and so is a second order differential operator, which may also be called the k-trace of D^2u. Especially, it is easily to see that the N-Hessian is the Monge-Ampére operator and that the 1-Hessian is the well known classical Laplace operator. Hence, the k-Hessian operators form a discrete collection of partial differential operators which includes both the Laplace and the Monge-Ampére operator.

In this paper we study the existence of radial solutions for the following Hessian equation

$$S_k^{1/k}\left(\lambda\left(D^2u\right)\right) = p\left(|x|\right)h\left(u\right) \text{ in } \mathbb{R}^N , \tag{1.1}$$

and system

$$\left\{\begin{array}{l} S_k^{1/k}\left(\lambda\left(D^2u\right)\right) = p\left(|x|\right)f\left(u,v\right) \text{ in } \mathbb{R}^N , \\ S_k^{1/k}\left(\lambda\left(D^2v\right)\right) = q\left(|x|\right)g\left(u,v\right) \text{ in } \mathbb{R}^N , \end{array}\right. \tag{1.2}$$

where $k \in \{1, 2, ..., N\}$, the continuous functions p, $q : [0, \infty) \to (0, \infty)$, $h : [0, \infty) \to [0, \infty)$ and f, $g : [0, \infty) \times [0, \infty) \to [0, \infty)$ satisfy some of the conditions:

(P1) p, q is a spherically symmetric function (i.e. $p(x) = p(|x|)$, $q(x) = q(|x|)$);

(P2) $r^{N+\frac{N}{k}-2}p^k(r)$ is nondecreasing for large r;

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(P3) $r^{N+\frac{N}{k}-2}\left[p^{k}\left(r\right)+q^{k}\left(r\right)\right]$ is nondecreasing for large r;

(C1) h is monotone non-decreasing, $h(0)=0$ and $h\left(s\right)>0$ for all $s>0$;

(C2) f, g are monotone non-decreasing in each variable, $f(0,0)=g\left(0,0\right)=0$ and $f(s,t)>0$, $g\left(s,t\right)>0$ for all $s,t>0$;

(C3) $\int_{1}^{\infty}\frac{1}{\sqrt[k+1]{(k+1)H(t)}}dt=\infty$ for $H(t)=\int_{0}^{t}h^{k}(z)dz$;

(C4) $\int_{1}^{\infty}\frac{1}{\sqrt[k+1]{(k+1)F(t)}}dt=\infty$ for $F(t)=\int_{0}^{t}\left(f^{k}(z,z)+g^{k}(z,z)\right)dz$.

The properties of the k-Hessian operator was well discussed in a numerous papers written as a first author by Ivochkina (see [7]-[10] and others). Moreover, this operator appear as an object of investigation by many remarkable geometers. For example, Viaclovsky (see [17], [18]) observed that the $k-$Hessian operator is an important class of fully nonlinear operators which is closely related to a geometric problem of the type (1.1), where we cite the work of Bao-Ji-Li [2] for a more detailed discussion. Moreover, equation (1.1) arises via the study of the quasilinear parabolic problem (see for example the introduction of Moll-Petitta [15]). In the present work we will limit ourselves to the development of mathematical theory for (1.1) and (1.2). The main difficulty in investigating problems, such as (1.1) or (1.2), in which appear the k-Hessian operator is related to the fact that their properties change depending on the subset of C^{2} from where the solution is taken. Our main objective here is to find functions in C^{2} that are strictly k-convex and verifies the problems (1.1), (1.2), where by strictly k-convex function u we mean that all eigenvalues $\lambda_{1},...,\lambda_{N}$ of the symmetric matrix $D^{2}u$ are in the so called Gårdding open cone Γ_{k} which is defined by

$$\Gamma_{k}\left(N\right)=\left\{\lambda\in\mathbb{R}^{N}\left|S_{1}\left(\lambda\right)>0,....,S_{k}\left(\lambda\right)>0\right.\right\}$$

In the next we adopt the notation from Bao-Li [12] for the space of all admissible functions

$$\Phi^{k}\left(\mathbb{R}^{N}\right):=\left\{u\in C^{2}\left(\mathbb{R}^{N}\right)\left|\lambda\in\Gamma_{k}\left(N\right)\text{ for all }x\in\mathbb{R}^{N}\right.\right\}.$$

In our direction, there are some recently papers resolving existence for blow-up solutions of (1.1) and (1.2). Here we wish to mention the works of Bao-Ji-Li [2], Jacobsen [11], Bao-Li [12], Lazer and McKenna [14, (the case $k=N$)], Salani [16] and Zhang-Zhou [19] which will be useful in our proofs. It is interesting to note that in our results the dimension of the space $\mathbb{R}^{N}$ affect the properties of the solution of the equation and system which in the case of the classical Laplace operator and the Monge-Ampére operator this condition doesn't appear in any works.

Motivated by the recent work of Zhang-Zhou [19] we are interested in proving the following theorems:

Theorem 1. *Let $k\in\{1,2,...,[N/2]\}$ if N is odd or $k\in\{1,2,...,[N/2]-1\}$ if N is even. Suppose that (P1), (P2), (C1), (C3) are satisfied. If there exists a positive number ε such that*

$$\int_{0}^{\infty}t^{1+\varepsilon+\frac{2(k-1)}{k+1}}\left(p\left(t\right)\right)^{\frac{2k}{k+1}}dt<\infty, \tag{1.3}$$

then system (1.1) has a nonnegative nontrivial radial bounded solution $u\in\Phi^{k}\left(\mathbb{R}^{N}\right)$.

Theorem 2. *If p satisfy (P1) and f satisfy (C1), (C3), then the problem (1.1) has a nonnegative nontrivial entire radial solution $u\in\Phi^{k}\left(\mathbb{R}^{N}\right)$. Suppose furthermore that (P2) holds. If p satisfies*

$$\int_{0}^{\infty}\left(\frac{k}{t^{N-k}}\int_{0}^{t}\frac{s^{N-1}}{C_{N-1}^{k-1}}p^{k}\left(s\right)ds\right)^{1/k}dt=\infty, \tag{1.4}$$

then any nonnegative nontrivial radial solution $u\in\Phi^{k}\left(\mathbb{R}^{N}\right)$ of (1.1) is large. Conversely, if (1.1) has a nonnegative entire large radial solution $u\in\Phi^{k}\left(\mathbb{R}^{N}\right)$, then one or both of the

following

$$(1.5)\qquad \begin{array}{ll} 1. & \int_0^\infty t^{1+\varepsilon+\frac{2(k-1)}{k+1}}\left(p\left(t\right)\right)^{\frac{2k}{k+1}}dr=\infty \text{ for every } \varepsilon>0\,; \\ 2. & k\in\left\{\left[N/2\right]+1,...,N\right\} \text{ if } N \text{ is odd or } k\in\left\{\left[N/2\right],...,N\right\} \text{ if } N \text{ is even,} \end{array}$$

hold.

Regarding existence of solution to (1.2), we have the following results.

Theorem 3. *Let $k\in\{1,2,...,[N/2]\}$ if N is odd or $k\in\{1,2,...,[N/2]-1\}$ if N is even. Suppose that (P1), (P3), (C2), (C4) are satisfied. If there exists a positive number ε such that*

$$\int_0^\infty t^{1+\varepsilon+\frac{2(k-1)}{k+1}}\left(p^k\left(t\right)+q^k\left(t\right)\right)^{\frac{2}{k+1}}dt<\infty\ , \tag{1.6}$$

then system (1.2) has a nonnegative nontrivial radial bounded solution $(u,v)\in\Phi^k\left(\mathbb{R}^N\right)\times\Phi^k\left(\mathbb{R}^N\right)$.

Theorem 4. *If p, q satisfy (P1) and f, g satisfy (C2), (C4), then the problem (1.1) has a nonnegative nontrivial entire radial solution. Suppose furthermore that (P3) holds. If p satisfies*

$$\int_0^\infty\left(\frac{k}{t^{N-k}}\int_0^t\frac{s^{N-1}}{C_{N-1}^{k-1}}p^k\left(s\right)ds\right)^{1/k}dt=\infty \text{ and } \int_0^\infty\left(\frac{k}{t^{N-k}}\int_0^t\frac{s^{N-1}}{C_{N-1}^{k-1}}q^k\left(s\right)ds\right)^{1/k}dt=\infty, \tag{1.7}$$

then any nonnegative nontrivial solution $(u,v)\in\Phi^k\left(\mathbb{R}^N\right)\times\Phi^k\left(\mathbb{R}^N\right)$ of (1.2) is large. Conversely, if (1.2) has a nonnegative entire large radial solution $(u,v)\in\Phi^k\left(\mathbb{R}^N\right)\times\Phi^k\left(\mathbb{R}^N\right)$, then one or both of the following

$$(1.8)\qquad \begin{array}{ll} 1. & \int_0^\infty t^{1+\varepsilon+\frac{2(k-1)}{k+1}}\left(p^k\left(t\right)+q^k\left(t\right)\right)^{\frac{2}{k+1}}dr=\infty \text{ for every } \varepsilon>0\,; \\ 2. & k\in\left\{\left[N/2\right]+1,...,N\right\} \text{ if } N \text{ is odd or } k\in\left\{\left[N/2\right],...,N\right\} \text{ if } N \text{ is even,} \end{array}$$

hold.

For the readers' convenience, we recall the radial form of the k-Hessian operator.

Remark 5. *(see, for example,* [12], [16]*) If $u:\mathbb{R}^N\to\mathbb{R}$ is radially symmetric then a calculation show*

$$S_k\left(\lambda\left(D^2u\left(r\right)\right)\right)=r^{1-N}C_{N-1}^{k-1}\left[\frac{r^{N-k}}{k}\left(u^{'}\left(r\right)\right)^k\right]^{'}\ ,$$

where the prime denotes differentiation with respect to $r=|x|$ and $C_{N-1}^{k-1}=(N-1)!/\left[(k-1)!(N-k)!\right]$.

2. Proofs of the main results

In this section we give the proofs of Theorems 1 - 4. The main references for proving Theorems 1 - 2 is the work of Lair [13] and Delanoë [5] see also Afrouzi-Shokooh [1].
Proof of the Theorem 1. Assume that (1.3) holds. We prove the existence of $w\in\Phi^k\left(\mathbb{R}^N\right)$ to the problem

$$S_k^{1/k}\left(\lambda\left(D^2w\left(|x|\right)\right)\right)=p\left(|x|\right)h\left(w\left(|x|\right)\right) \text{ in } \mathbb{R}^N. \tag{2.1}$$

Observe that we can rewrite (2.1) as follows:

$$\left[\frac{r^{N-k}}{k}\left(w^{'}\left(r\right)\right)^k\right]^{'}=\frac{r^{N-1}}{C_{N-1}^{k-1}}p^k\left(r\right)h^k\left(w\left(r\right)\right),\ r=|x|\,.$$

Then radial solutions of (2.1) are any solution w of the integral equation

$$w(r) = 1 + \int_0^r \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} p^k(s) h^k(w(s)) \, ds \right)^{1/k} dt.$$

To establish a solution to this problem, we use successive approximation. Define sequence $\{w^m\}^{m \geqslant 1}$ on $[0, \infty)$ by

$$\begin{cases} w^0 = 1, \ r \geqslant 0, \\ w^m(r) = 1 + \int_0^r \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} p^k(s) h^k\left(w^{m-1}(s)\right) ds \right)^{1/k} dt. \end{cases}$$

We remark that, for all $r \geqslant 0$ and $m \in N$

$$w^m(r) \geqslant 1.$$

Moreover, proceeding by induction we conclude $\{w^m\}^{m \geqslant 1}$ are non-decreasing sequence on $[0, \infty)$. We note that $\{w^m\}^{m \geqslant 1}$ satisfies

$$\left\{ \frac{r^{N-k}}{k} \left[\left(w^m(r)\right)' \right]^k \right\}' = \frac{r^{N-1}}{C_{N-1}^{k-1}} p^k(r) h^k\left(w^{m-1}(r)\right).$$

By the monotonicity of $\{w^m\}^{m \geqslant 1}$ we have the inequalities

$$\left\{ \frac{r^{N-k}}{k} \left[\left(w^m(r)\right)' \right]^k \right\}' = \frac{r^{N-1}}{C_{N-1}^{k-1}} p^k(r) h^k\left(w^{m-1}(r)\right) \leqslant \frac{r^{N-1}}{C_{N-1}^{k-1}} p^k(r) h^k\left(w^m(r)\right). \tag{2.2}$$

Choose $R > 0$ so that $r^{N+\frac{N}{k}-2} p^k(r)$ are non-decreasing for $r \geqslant R$. We are now ready to show that $w^m(R)$ and $(w^m(R))'$, both of which are nonnegative, are bounded above independent of m. To do this, let

$$\phi^R = \max\{p^k(r) : 0 \leqslant r \leqslant R\}.$$

Using this and the fact that $(w^m)' \geqslant 0$, we note that (2.2) yields

$$\begin{aligned} r^{N-k} \left[\left(w^m(r)\right)' \right]^{k-1} \left(w^m(r)\right)'' &\leqslant \frac{N-k}{k} r^{N-k-1} \left[\left(w^m(r)\right)' \right]^k + r^{N-k} \left[\left(w^m(r)\right)' \right]^{k-1} \left(w^m(r)\right)'' \\ &\leqslant \phi_1^R \frac{r^{N-1}}{C_{N-1}^{k-1}} h^k\left(w^m(r)\right), \end{aligned}$$

and moreover

$$r^{N-k} \left[\left(w^m(r)\right)' \right]^{k-1} \left(w^m(r)\right)'' \leqslant \phi^R \frac{r^{N-k+k-1}}{C_{N-1}^{k-1}} h^k\left(w^m(r)\right) \leqslant R^{k-1} \phi^R \frac{r^{N-k}}{C_{N-1}^{k-1}} h^k\left(w^m(r)\right),$$

from which we have

$$\left[\left(w^m(r)\right)' \right]^{k-1} \left(w^m(r)\right)'' \leqslant R^{k-1} \phi^R \frac{1}{C_{N-1}^{k-1}} h^k\left(w^m(r)\right).$$

Multiply this by $\left(w^m(r)\right)'$ we obtain

$$\left\{ \left[\left(w^m(r)\right)' \right]^{k+1} \right\}' \leqslant \frac{(k+1) R^{k-1} \phi^R}{C_{N-1}^{k-1}} h^k\left(w^m(r)\right) \left(w^m(r)\right)'. \tag{2.3}$$

Integrate (2.3) from 0 to r to get

(2.4)

$$\left[\left(w^m(r)\right)' \right]^{k+1} \leqslant \frac{(k+1) R^{k-1} \phi^R}{C_{N-1}^{k-1}} \int_0^r h^k\left(w^m(s)\right) \left(w^m(s)\right)' ds = \frac{(k+1) R^{k-1} \phi^R}{C_{N-1}^{k-1}} \int_1^{w^m(r)} h^k(s) \, ds$$

for $0 \leqslant r \leqslant R$, which yields

$$\int_1^{w^m(R)} \left[\int_1^t h^k(s)\, ds\right]^{-1/(k+1)} dt \leqslant \sqrt[k+1]{\frac{(k+1)\,\phi^R}{C_{N-1}^{k-1}}} \cdot R^{\frac{2k}{k+1}}.$$

It follows from the above relation and by the assumption (C2) that $w_1^m(R)$ is bounded above independent of m. Using this fact in (2.4) shows that the same is true of $(w^m(R))'$. Thus, the sequences $w^m(R)$ and $(w^m(R))'$ are bounded above independent of m.

Finally, we show that the non-decreasing sequences w^m is bounded for all $r \geqslant 0$ and all m. Multiplying the equation (2.2) by $r^{N+\frac{N}{k}-2}\left(w^m(r)\right)'$, we get

$$\left\{\left[r^{\frac{N}{k}-1}(w^m(r))'\right]^{k+1}\right\}' = \frac{k+1}{C_{N-1}^{k-1}} p^k(r)\, h^k(w^m(r)) r^{N+\frac{N}{k}-2}(w^m(r))'. \tag{2.5}$$

Integrating from R to r gives

$$\left[r^{\frac{N}{k}-1}(w^m(r))'\right]^{k+1} = \left[R^{\frac{N}{k}-1}(w^m(R))'\right]^{k+1} + \frac{k+1}{C_{N-1}^{k-1}} \int_R^r p^k(s)\, h^k(w^m(s)) s^{N+\frac{N}{k}-2}(w^m(s))'\, ds,$$

for $r \geqslant R$. Noting that, by the monotonicity of $s^{N+\frac{N}{k}-2}p^k(s)$ for $r \geqslant s \geqslant R$, we get

$$\left[r^{\frac{N}{k}-1}(w^m(r))'\right]^{k+1} \leqslant C + \frac{k+1}{C_{N-1}^{k-1}} r^{N+\frac{N}{k}-2} p^k(r)\, H(w^m(r))$$

where $C = \left[R^{\frac{N}{k}-1}(w^m(R))'\right]^{k+1}$, which yields

$$r^{\frac{N}{k}-1}(w^m(r))' \leqslant C^{\frac{1}{k+1}} + \left(\frac{k+1}{C_{N-1}^{k-1}}\right)^{\frac{1}{k+1}} r^{\frac{N}{k}-\frac{2}{k+1}} p^{\frac{k}{k+1}}(r)\, H^{\frac{1}{k+1}}(w^m(r))$$

or, equivalently

$$(w^m(r))' \leqslant C^{\frac{1}{k+1}} r^{1-\frac{N}{k}} + \left(\frac{k+1}{C_{N-1}^{k-1}}\right)^{\frac{1}{k+1}} r^{1-\frac{2}{k+1}} p^{\frac{k}{k+1}}(r)\, H^{\frac{1}{k+1}}(w^m(r))$$

and hence

$$\frac{d}{dr}\int_{w^m(R)}^{w^m(r)} [H(t)]^{-1/(k+1)}\, dt \leqslant C^{\frac{1}{k+1}} r^{1-\frac{N}{k}} H^{-\frac{1}{k+1}}(w^m(r)) + \left(\frac{k+1}{C_{N-1}^{k-1}}\right)^{\frac{1}{k+1}} \left(r^{k-1}p^k(r)\right)^{\frac{1}{k+1}}. \tag{2.6}$$

Inequality (2.6) combined with

$$\begin{aligned} \frac{1}{\sqrt{2}}\sqrt{2\cdot\left(s^{k-1}p^k(s)\right)^{\frac{2}{k+1}}} &= \frac{1}{\sqrt{2}}\sqrt{2\cdot s^{\frac{1+\varepsilon}{2}}\left(s^{k-1}p^k(s)\right)^{\frac{2}{k+1}} s^{\frac{1-\varepsilon}{2}}} \\ &\leqslant \frac{1}{\sqrt{2}}\left[s^{1+\varepsilon}\left(s^{k-1}p^k(s)\right)^{\frac{2}{k+1}} + s^{-1-\varepsilon}\right] \end{aligned}$$

gives

$$\begin{aligned} \textstyle\int_{w^m(R)}^{w^m(r)} [H(t)]^{-1/(k+1)}\, dt &\leqslant C^{\frac{1}{k+1}} \textstyle\int_R^r t^{1-\frac{N}{k}} H^{-\frac{1}{k+1}}(w^m(t))\, dt \\ &\quad + \tfrac{1}{\sqrt{2}}\left(\tfrac{k+1}{C_{N-1}^{k-1}}\right)^{\frac{1}{k+1}}\left[\textstyle\int_R^r t^{1+\varepsilon+\frac{2(k-1)}{k+1}}(p(t))^{\frac{2k}{k+1}}\, dt + \textstyle\int_R^r t^{-1-\varepsilon} dt\right] \\ &\leqslant C^{\frac{1}{k+1}} H^{-\frac{1}{k+1}}(w^m(R)) \textstyle\int_R^r t^{1-\frac{N}{k}}\, dt \\ &\quad + \tfrac{1}{\sqrt{2}}\left(\tfrac{k+1}{C_{N-1}^{k-1}}\right)^{\frac{1}{k+1}}\left[\textstyle\int_R^r t^{1+\varepsilon+\frac{2(k-1)}{k+1}}(p(t))^{\frac{2k}{k+1}}\, dt + \tfrac{1}{\varepsilon R^{\varepsilon}}\right]. \end{aligned}$$

The above relation is needed in proving the bounded of the function $\{w^m\}^{m\geqslant 1}$ in the following. Indeed, since for each $\varepsilon > 0$ the right side of this inequality is bounded independent of m

(note that $w^m(t) \geqslant 1$), so is the left side and hence, in light of (C2), the sequence $\{w^m\}^{m\geqslant 1}$ is a bounded sequence and so $\{w^m\}^{m\geqslant 1}$ are bounded sequence. Thus $\{w^m\}^{m\geqslant 1} \to w$ as $m \to \infty$ and the limit functions w are positive entire bounded solutions of equation (2.1).
Proof of the Theorem 2. We know that for any $a_1 > 0$ a solution of

$$v(r) = a_1 + \int_0^r \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} p^k(s) h^k(v(s))\, ds \right)^{1/k} dt,$$

exists, at least, small r. Since $v' \geqslant 0$, the only way that the solution can become singular at R is for $v(r) \to \infty$ as $r \longrightarrow \infty$. Thus, we can show that, for each $R > 0$, there exists $C_R > 0$ so that $v(R) \leqslant C_R$, we have existence. To this end, let $M_R = \max\{p(r) | 0 \leqslant r \leqslant R\}$ and consider the equation

$$w(r) = a_2 + M_R \int_0^r \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} h^k(v(s))\, ds \right)^{1/k} dt$$

where $a_2 > a_1$. We next observe that the solution to this equation exists for all $r \geqslant 0$ and of course, it is a solution to $S_k^{1/k}\left(\lambda\left(D^2 w(r)\right)\right) = M_R h(w)$ on $\mathbb{R}^N$ which is treated in [12, (Theorem 1.1, p. 177)]. We now show that $v(r) \leqslant w(r)$ for all $0 \leqslant r \leqslant R$ and hence we conclude the proof of existence. Clearly $v(0) < w(0)$ so that $v(r) < w(r)$ for at least all r near zero. Let

$$r_0 = \sup\{r | v(s) < w(s) \text{ for all } s \in [0, r]\}.$$

If $r_0 = R$, then we are done. Thus assume that $r_0 < R$. It follows from assumption $a_2 > a_1$ that

$$\begin{aligned} v(r_0) &= a_1 + \int_0^{r_0} \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} p^k(s) h^k(v(s))\, ds \right)^{1/k} dt \\ &< a_2 + M_R \int_0^{r_0} \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} h^k(v(s))\, ds \right)^{1/k} dt = w(r_0). \end{aligned}$$

Thus there exists $\varepsilon > 0$ so that $v(r) < w(r)$ for all $[0, r+\varepsilon)$, contradicting the definition of r_0. Thus we conclude that $v < w$ on $[0, R]$ for all $R > 0$ and hence v is a nontrivial entire solution of (1.1). Now let u be any nonnegative nontrivial entire solution of (1.1) and suppose p satisfies

$$\int_0^\infty \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} p^k(s)\, ds \right)^{1/k} dt = \infty.$$

Since u is nontrivial and non-negative, there exists $R > 0$ so that $u(R) > 0$. On the other hand since $u' \geqslant 0$, we get $u(r) \geqslant u(R)$ for $r \geqslant R$ and thus from

$$u(r) = u(0) + \int_0^r \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} p^k(s) h^k(u(s))\, ds \right)^{1/k} dt,$$

since u will satisfy that equation for all $r \geqslant 0$, we get

$$\begin{aligned} u(r) &= u(0) + \textstyle\int_0^r \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} p^k(s) h^k(u(s))\, ds \right)^{1/k} dt \\ &\geqslant u(R) + h(u(R)) \textstyle\int_R^r \left(\frac{k}{t^{N-k}} \int_R^t \frac{s^{N-1}}{C_{N-1}^{k-1}} p^k(s)\, ds \right)^{1/k} dt \to \infty \text{ as } r \to \infty. \end{aligned}$$

Conversely, assume that h satisfy (C1), (C3) and that w is a nonnegative entire large solution of (1.1). Note also, that w satisfies

$$\left[\frac{r^{N-k}}{k} \left(w'(r) \right)^k \right]' = \frac{r^{N-1}}{C_{N-1}^{k-1}} p^k(r) h^k(w(r)).$$

Using the monotonicity of $r^{N+\frac{N}{k}-2}p(r)$ we can apply similar arguments used in obtaining Theorem 1 to get

$$(w(r))' \leqslant C^{\frac{1}{k+1}} r^{1-\frac{N}{k}} + \left(\frac{k+1}{C_{N-1}^{k-1}}\right)^{\frac{1}{k+1}} r^{1-\frac{2}{k+1}} p^{\frac{k}{k+1}}(r) H^{\frac{1}{k+1}}(w(r)),$$

which we may rewrite as
(2.7)

$$\begin{aligned}
\int_{w(R)}^{w(r)} [H(t)]^{-1/(k+1)} dt &\leqslant C^{\frac{1}{k+1}} \int_R^r t^{1-\frac{N}{k}} H^{-\frac{1}{k+1}}(w(t)) dt \\
&+ \frac{1}{\sqrt{2}} \left[\int_R^r t^{1+\varepsilon+\frac{2(k-1)}{k+1}} (p(t))^{\frac{2k}{k+1}} dt + \int_R^r t^{-1-\varepsilon} dt\right] \\
&\leqslant C^{\frac{1}{k+1}} H^{-\frac{1}{k+1}}(w(R)) \int_R^r t^{1-\frac{N}{k}} dt \\
&+ \frac{1}{\sqrt{2}} \left(\frac{k+1}{C_{N-1}^{k-1}}\right)^{\frac{1}{k+1}} \left[\int_R^r t^{1+\varepsilon+\frac{2(k-1)}{k+1}} (p(t))^{\frac{2k}{k+1}} dt + \frac{1}{\varepsilon R^{\varepsilon}}\right] \\
&\leqslant C_R \int_R^r t^{1-\frac{N}{k}} dt + \frac{1}{\sqrt{2}} \left(\frac{k+1}{C_{N-1}^{k-1}}\right)^{\frac{1}{k+1}} \int_R^r t^{1+\varepsilon+\frac{2(k-1)}{k+1}} (p(t))^{\frac{2k}{k+1}} dt
\end{aligned}$$

where

$$C_R = C^{\frac{1}{k+1}} H^{-\frac{1}{k+1}}(w(R)) + \frac{1}{\sqrt{2}} \frac{1}{\varepsilon R^{\varepsilon}} \left(\frac{k+1}{C_{N-1}^{k-1}}\right)^{\frac{1}{k+1}}.$$

By taking $r \to \infty$ in (2.7) we obtain (1.5) since w is large and h satisfies (C3). These observations completes the proof of the theorem.

Proof of the Theorem 3 and 4. In order, to obtain the conclusion, combine the proof of **Theorem 1** and **2** with some technical results from [3] and [4].

□

References

[1] G. A. Afrouzi and S. Shokooh, *Large Solution of Quasilinear Elliptic Equations under the Keller-Osserman Condition*, International Journal of Mathematical Analysis, Volume 4, 2010, Number 42, Pages 2065 - 2074.

[2] J. Bao, X. Ji and H. Li, *Existence and nonexistence theorem for entire subsolutions of k-Yamabe type equations*, Journal of Differential Equations, Volume 253, Pages 2140–2160, 2012.

[3] D.-P. Covei, *Radial and nonradial solutions for a semilinear elliptic system of Schrodinger type*, Funkcialaj Ekvacioj - Serio Internacia, Volume 54, 2011, Pages 439-449.

[4] D.-P. Covei, *An existence result for a quasilinear system with gradient term under the Keller-Osserman conditions*, Turkish Journal of Mathematics, Volume 38, Pages 267-277, 2014.

[5] PH. Delanoë, *Radially symmetric boundary value problems for real and complex elliptic Monge-Ampère equations*, Journal of Differential Equations, Volume 58, Pages 318-344, 1985.

[6] N M Ivochkina, *The integral method of barrier functions and the Dirichlet problem for equations with operators of Monge-Ampere type*, Mat. Sb. 112 (154) (1980), 193-206; English transl. in Math. USSR Sb. 40 (1981).

[7] N M Ivochkina, *A description of the stability cones generated by differential operators of Monge-Ampere type*, Mathematics of the USSR-Sbornik, Volume 122, Number 164, 1983 (pp. 265-275; English transl. in Math. USSR Sb. 50, 1985).

[8] N M Ivochkina, *Solution of the Dirichlet problem for some equations of Monge-Ampère type,* Mathematics of the USSR-Sbornik, Volume 56, Number 2, 1987.

[9] N M Ivochkina, *Solution of the Dirichlet problem for curvature equations of order m*, Mathematics of the USSR-Sbornik, Volume 67, Number 2, 1989.

[10] N. M. Ivochkina and N. V. Filimonenkova, *On algebraic and geometric conditions in the theory of Hessian equations*, Journal of Fixed Point Theory and Applications, Volume 16, Pages 11–25, 2014.

[11] J. Jacobsen, *A Liouville-Gelfand Equation for k-Hessian Operators, Rocky Mountain Journal of Mathematics*, Volume 34, Number 2, Pages 665-684, 2004.

[12] X. Ji and J. Bao, *Necessary and sufficient conditions on solvability for Hessian inequalities*, Proceedings of the American Mathematical Society, Volume 138, Number 1, January 2010, Pages 175–188.

[13] A.V. Lair, *Large solutions of semilinear elliptic equations under the Keller–Osserman condition*, Journal of Mathematical Analysis and Applications, Volume 328, Issue 2, 15 April 2007, Pages 1247-1254.

[14] A. C. Lazer and P. J. McKenna, *On singular boundary value problems for the Monge-Ampére operator*, Journal of Mathematical Analysis and Applications, Volume 197, Issue 2, Pages 341–362, January 1996.

[15] S. Moll and F. Petitta, *Large solutions for nonlinear parabolic equations without absorption terms*, Journal of Functional Analysis, Volume 262, Pages 1566–1602, 2012.

[16] P. Salani, *Boundary blow-up problems for Hessian equations*, Manuscripta Mathematica, Volume 96, Pages 281 – 294, 1998.

[17] J.A. Viaclovsky, *Conformal geometry, contact geometry, and the calculus of variations*, Duke Mathematical Journal, Volume 101, Pages 283–316, 2000.

[18] J.A. Viaclovsky, *Estimates and existence results for some fully nonlinear elliptic equations on Riemannian manifolds*, Communications in Analysis and Geometry, Volume 10, Pages 815–846, 2002.

[19] Z. Zhang and S. Zhou, *Existence of entire positive k-convex radial solutions to Hessian equations and systems with weights*, Applied Mathematics Letters, Volume 50, December 2015, Pages 48–55.

Department of Applied Mathematics, The Bucharest University of Economic Studies, Piata Romana, 1st district, postal code: 010374, postal office: 22, Romania

E-mail address: coveidragos@yahoo.com